

# A kinetic Dirac interpolation theorem for homogeneous quasi-free fermions

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## Abstract

How much exchange energy can a homogeneous fermion gas concede beyond the Dirac value, and what controls the deviation? We answer this question completely within the gauge-invariant quasi-free class. The bound we obtain interpolates between the Dirac constant  $C_D(q)$ , attained at the unpolarized Fermi ball, and the fully spin-polarized constant  $C_D(1)$ , attained in the large-kinetic-excess regime. The interpolation parameter is the homogeneous kinetic ratio  $R = \tau/(K_D(q)\rho^{5/3}) \geq 1$ , which the Pauli principle forces above unity. The proof rests on two classical inputs: the Riesz rearrangement inequality, which bounds the exchange integral by the value at indicator-of-ball occupations, and the bathtub principle, which converts the kinetic budget into a spin-polarization constraint. A short Lyapunov-style interpolation between  $\ell^1$  and  $\ell^{5/3}$  then yields the explicit  $\sqrt{R}$  law and the small-excess slope at  $R = 1$ . The result isolates what the quasi-free framework alone can establish about the Lieb–Oxford problem.

## 1 Introduction

How does the exchange energy of a homogeneous fermion gas depend on its kinetic budget? The classical Dirac formula  $e_x = -C_D(q)\rho^{4/3}$  describes the unpolarized Fermi sea, but the same density supports a continuum of homogeneous quasi-free states with strictly larger kinetic energy. The natural question is whether, and how, the Dirac constant can degrade across that family. The Lieb–Oxford program identifies  $C_D(q)$  as the right benchmark constant for unpolarized matter; outside the quasi-free class, getting a sharp universal lower bound is a longstanding open problem.

This note answers the homogeneous quasi-free version of the question in closed form. We work with gauge-invariant quasi-free states in the thermodynamic limit, parametrized by occupation functions  $0 \leq n_\sigma(k) \leq 1$ . The Pauli principle forces the homogeneous kinetic ratio

$$R = \frac{\tau}{K_D(q)\rho^{5/3}}$$

to be at least one, with equality only at the unpolarized Fermi ball. Theorem 2.1 states that for the entire quasi-free class

$$e_x(n) \geq -C_D(q) \min\{\sqrt{R}, q^{1/3}\} \rho^{4/3},$$

which reduces to the Dirac bound  $-C_D(q)\rho^{4/3}$  at  $R = 1$  and saturates at the fully spin-polarized value  $-C_D(1)\rho^{4/3}$  for  $R \geq q^{2/3}$ . The mechanism is transparent: kinetic excess can only widen the exchange constant by spin polarization, and the Pauli phase-space constraint converts a

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kinetic budget into a permitted level of polarization. The two-page proof combines the Riesz rearrangement inequality with the bathtub principle and an  $\ell^p$  interpolation step.

The takeaway for the Lieb–Oxford problem is structural. The bound here is tight within the quasi-free framework; any further descent toward the Lieb–Oxford constant  $C_{\text{LO}}$  must therefore originate in genuine two-particle correlation, not in one-particle phase-space rearrangement.

**Plan of the paper.** Section 2 sets up notation and states the main theorem. Section 3 proves the one-spin exchange bound via Riesz rearrangement. Section 4 establishes the kinetic moment constraint via the bathtub principle. Section 5 assembles these ingredients into the proof of the main theorem and records the small-excess expansion.

## 2 Statement

Fix spin degeneracy  $q$ . A homogeneous gauge-invariant quasi-free fermion state in the thermodynamic limit is described by occupation functions

$$0 \leq n_\sigma(k) \leq 1, \quad \sigma = 1, \dots, q, \quad k \in \mathbb{R}^3.$$

Its spin densities and kinetic-energy densities are

$$\rho_\sigma = (2\pi)^{-3} \int_{\mathbb{R}^3} n_\sigma(k) dk, \quad \tau_\sigma = (2\pi)^{-3} \int_{\mathbb{R}^3} |k|^2 n_\sigma(k) dk.$$

Set

$$\rho = \sum_{\sigma=1}^q \rho_\sigma, \quad \tau = \sum_{\sigma=1}^q \tau_\sigma.$$

Let

$$K_D(q) = \frac{3}{5} (6\pi^2)^{2/3} q^{-2/3}, \quad C_D(q) = \frac{3}{4} \left(\frac{6}{\pi}\right)^{1/3} q^{-1/3}.$$

Define the homogeneous kinetic ratio

$$R = \frac{\tau}{K_D(q)\rho^{5/3}}.$$

By the Pauli principle  $R \geq 1$ .

The exchange energy density of the quasi-free state is

$$e_x(n) = -\frac{1}{2} \sum_{\sigma=1}^q (2\pi)^{-6} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{4\pi}{|k-\ell|^2} n_\sigma(k) n_\sigma(\ell) dk d\ell.$$

**Theorem 2.1** (Quasi-free kinetic Dirac interpolation). *For every homogeneous quasi-free fermion state,*

$$e_x(n) \geq -C_D(q) \min\{\sqrt{R}, q^{1/3}\} \rho^{4/3}.$$

*In particular, when  $R \downarrow 1$ ,*

$$e_x(n) \geq -C_D(q)\rho^{4/3} - \frac{1}{2}C_D(q)(R-1)\rho^{4/3} + O((R-1)^2)\rho^{4/3}.$$

*Thus the lower bound has the Dirac constant as its small-excess kinetic limit.*

*Remark 2.2.* The statement is positive, not a no-go result. Across the homogeneous quasi-free class, the Dirac constant follows from two ingredients alone: the Pauli phase-space constraint and the kinetic ratio. The large- $R$  saturation value  $C_D(1)$  is the fully spin-polarized exchange constant; it does not descend to the Lieb–Oxford constant  $C_{\text{LO}}$ , because quasi-free states carry no correlation energy beyond exchange.

### 3 Exchange rearrangement

**Lemma 3.1** (One-spin exchange maximization). *Let  $0 \leq n(k) \leq 1$  and*

$$\rho_n = (2\pi)^{-3} \int n(k) dk.$$

Then

$$\frac{1}{2}(2\pi)^{-6} \iint \frac{4\pi}{|k - \ell|^2} n(k)n(\ell) dk d\ell \leq C_D(1)\rho_n^{4/3}.$$

Equality holds for the indicator of a ball, up to translation.

*Proof.* The kernel  $|k - \ell|^{-2}$  is radially decreasing. By the Riesz rearrangement inequality, the left side is maximized, among all functions  $0 \leq n \leq 1$  with fixed integral, by the symmetric decreasing rearrangement  $n^* = \mathbf{1}_{B_{k_n}}$ , where

$$\frac{|B_{k_n}|}{(2\pi)^3} = \rho_n, \quad k_n = (6\pi^2\rho_n)^{1/3}.$$

For this ball,

$$\iint_{B_{k_n} \times B_{k_n}} \frac{4\pi}{|k - \ell|^2} dk d\ell = 16\pi^3 k_n^4.$$

Therefore

$$\begin{aligned} \frac{1}{2}(2\pi)^{-6} 16\pi^3 k_n^4 &= \frac{k_n^4}{8\pi^3} \\ &= \frac{3}{4} \left(\frac{6}{\pi}\right)^{1/3} \rho_n^{4/3} = C_D(1)\rho_n^{4/3}. \end{aligned}$$

□

### 4 Kinetic moment constraint

**Lemma 4.1** (Spin-density moment bound). *For each spin component,*

$$\tau_\sigma \geq K_D(1)\rho_\sigma^{5/3}, \quad K_D(1) = \frac{3}{5}(6\pi^2)^{2/3}.$$

Consequently, with  $a_\sigma = \rho_\sigma/\rho$ ,

$$\sum_{\sigma=1}^q a_\sigma^{5/3} \leq Rq^{-2/3}.$$

*Proof.* Among all  $0 \leq n \leq 1$  with fixed integral, the moment  $\int |k|^2 n(k) dk$  is minimized by filling a ball centered at the origin. This is the bathtub principle. Thus

$$\tau_\sigma \geq (2\pi)^{-3} \int_{|k| \leq (6\pi^2\rho_\sigma)^{1/3}} |k|^2 dk = K_D(1)\rho_\sigma^{5/3}.$$

Summing over  $\sigma$  gives

$$\tau \geq K_D(1)\rho^{5/3} \sum_{\sigma} a_\sigma^{5/3}.$$

Since  $K_D(q) = K_D(1)q^{-2/3}$ , this is equivalent to

$$\sum_{\sigma} a_\sigma^{5/3} \leq Rq^{-2/3}.$$

□

## 5 Proof of the theorem

*Proof of Theorem 2.1.* By Theorem 3.1,

$$-e_x(n) \leq C_D(1) \sum_{\sigma=1}^q \rho_\sigma^{4/3} = C_D(1) \rho^{4/3} \sum_{\sigma=1}^q a_\sigma^{4/3}.$$

The elementary interpolation inequality between  $\ell^1$  and  $\ell^{5/3}$  gives

$$\sum_{\sigma=1}^q a_\sigma^{4/3} \leq \left( \sum_{\sigma=1}^q a_\sigma^{5/3} \right)^{1/2}, \quad \sum_{\sigma} a_\sigma = 1, \quad a_\sigma \geq 0.$$

Indeed,

$$\|a\|_{4/3} \leq \|a\|_1^{3/8} \|a\|_{5/3}^{5/8},$$

and raising to the 4/3 power gives the displayed inequality. Using Theorem 4.1,

$$\sum_{\sigma=1}^q a_\sigma^{4/3} \leq \sqrt{R} q^{-1/3}.$$

Since always

$$\sum_{\sigma=1}^q a_\sigma^{4/3} \leq 1,$$

we obtain

$$-e_x(n) \leq C_D(1) \rho^{4/3} \min\{\sqrt{R} q^{-1/3}, 1\}.$$

Finally  $C_D(q) = C_D(1) q^{-1/3}$ , and hence

$$e_x(n) \geq -C_D(q) \min\{\sqrt{R}, q^{1/3}\} \rho^{4/3}.$$

The expansion at  $R = 1$  follows from

$$\sqrt{R} = 1 + \frac{1}{2}(R - 1) + O((R - 1)^2).$$

□

*Remark 5.1.* The equality case at  $R = 1$  is the unpolarized Fermi ball:

$$\rho_1 = \cdots = \rho_q = \rho/q, \quad n_\sigma = \mathbf{1}_{\{|k| \leq (6\pi^2 \rho/q)^{1/3}\}}.$$

The theorem also clarifies the source of difficulty in the correlated Lieb problem. In the quasi-free class, kinetic excess inflates the exchange constant only through spin polarization, and the inflation obeys the explicit  $\sqrt{R}$  law of Theorem 2.1. Any descent from the Dirac constant toward  $C_{\text{LO}}$  therefore requires genuine two-particle correlation; it cannot arise from one-particle phase-space rearrangement alone.